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1969 J. Phys. A: Gen. Phys. 2 605

(http://iopscience.iop.org/0022-3689/2/5/013)

A new proof of superposition of dressed particles in plasma kinetic theory

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MS. received 23rd August 1968

Abstract. A new proof of the superposition of dressed particles in plasma kinetic theory is given by using the generalized stochastic equation for the conditional probability density of one particle given the position in phase space of another particle. The conditional probability density of *any* particle given a *specific* particle is shown to satisfy the Vlasov equation from which a Markov-type integral results. The superposition principle is established without the explicit introduction of test particles. Relations to Rostoker's results are discussed.

1. Introduction

The concept of 'dressed particles' has been proved to be very useful in many plasma problems (Montgomery and Tidman 1964). A 'dressed particle', as introduced by Rostoker (1961), is a particle which consists of a charge and the associated polarization cloud. In many cases, the collective behaviour of a plasma can be considered as a superposition of the individual behaviour of a collection of these dressed particles that are assumed to be uncorrelated. The electric field at a point $\boldsymbol{x}$ due to a dressed particle at position $\boldsymbol{x}'$ with velocity $\boldsymbol{v}'$ can be written as (Montgomery and Tidman 1964)

$$\boldsymbol{E}(\boldsymbol{x}, X') = \frac{e}{2\pi^2}\int\frac{\mathrm{d}\boldsymbol{k}\ \mathrm{i}\boldsymbol{k}\exp\{\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{x}-\boldsymbol{x}')\}}{k^2\epsilon^+(\boldsymbol{k}, -\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{v}')} \tag{1}$$

where $X' = (\boldsymbol{x}', \boldsymbol{v}')$ is the phase point of the dressed particle and

$$\epsilon^+(\boldsymbol{k}, p) = 1 - \mathrm{i}\,\frac{\omega_\mathrm{p}^2}{k^2}\,\boldsymbol{k}\,.\int\frac{\partial f_1}{\partial \boldsymbol{v}}\,\frac{\mathrm{d}\boldsymbol{v}}{p+\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{v}} \tag{2}$$

is the dielectric constant of the plasma. f_1 is the one-body distribution function and ω_p is the plasma frequency. Equation (1) can be used to compute the expectation value of the electric field as well as the correlation function for electric field fluctuations in the plasma. The results can then be used to derive the plasma kinetic equation (Montgomery and Tidman 1964). The justification of using such a scheme has been given by Rostoker (1964 a, b). He considers a test-particle problem and, by complicated manipulations, relates the results to the superposition principle of dressed particles.

In this paper an alternative and simpler proof will be given for this principle. Physically, we note that the concept of the polarization cloud about a charge is essentially a statistical one, since it involves the collection of other particles that produce this polarization cloud. Therefore it is natural to introduce the conditional probability density $G(X_2, t_2|X_1, t_1)$ of having any particle at (X_2, t_2) given particle 1 at (X_1, t_1). With the help of this function, the superposition principle can be proved without the explicit introduction of the test particle.

2. Derivation

Let us consider a fully ionized plasma consisting of electrons moving in a uniform background of infinite-mass positive ions. There are N electrons in a volume V. The interaction between two particles is through the Coulomb potential $\phi(|\boldsymbol{x}_i - \boldsymbol{x}_j|)$. Both N

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and V are assumed to approach infinity with the ratio $n_0 = N/V$ being the finite average particle number density. In the following, for the plasma case, $1/n_0$ is assumed to be a small quantity (Montgomery and Tidman 1964).

We define the two-particle–two-time probability density $D_{ij}(X, t; X', t')$ as the joint probability density of finding particle i at phase point $X(\boldsymbol{x}, \boldsymbol{v})$ at time t and particle j at phase point $X'(\boldsymbol{x}', \boldsymbol{v}')$ at time t'. $D_{ij}(X, t|X', t')$ will denote the conditional probability density of finding the ith particle at (X, t) given that the jth particle is at (X', t'). For indistinguishable particles the functions D_{ij} are identical for $i \neq j$; $i, j = 1, 2, ..., N$.

Let us write

$$D_i(X, t) = \int D_{ij}(X, t; X', t')\,\mathrm{d}X' = \frac{1}{V} f_1(X, t) \tag{3}$$

where $D_i(X, t)$ is the one-particle probability density function. In addition, since the particles are indistinguishable, we can write

$$D_{ij}(X, t|X', t') = \frac{1}{V}\,\{g(X, t|X', t') + f_1(X, t)\}, \qquad i \neq j \tag{4}$$

where the second term on the right-hand side is the probability density of finding a particle at (X, t) without the influence of the presence of a particle at (X', t'). The first term is the effect due to the presence of this particle at (X', t').

By considering the generalized stochastic equation for this plasma system, and applying the cluster expansion, it can be shown that the function g satisfies the following equation (So 1967, So and Yeh 1968):

$$\begin{aligned}
&\frac{\partial g(X_2, t_2|X_1, t_1)}{\partial t_2} + \boldsymbol{v}_2 \cdot \frac{\partial g(X_2, t_2|X_1, t_1)}{\partial \boldsymbol{x}_2} \\
&-\frac{n_0}{m}\frac{\partial f_1(X_2, t_2)}{\partial \boldsymbol{v}_2} \cdot \int \frac{\partial \phi(|\boldsymbol{x}_2 - \boldsymbol{x}_3|)}{\partial \boldsymbol{x}_2}\, g(X_3, t_2|X_1, t_1)\,\mathrm{d}X_3 \\
&= \frac{1}{m}\frac{\partial f_1(X_2, t_2)}{\partial \boldsymbol{v}_2} \cdot \frac{\partial \phi(|\boldsymbol{x}_2 - \boldsymbol{x}_1 - \boldsymbol{v}_1(t_2 - t_1)|)}{\partial \boldsymbol{x}_2}
\end{aligned} \tag{5}$$

where m is the mass of an electron.

Since our purpose is to prove the superposition principle, the derivation of (5) will not be given here. It is interesting to note that a similar equation for the joint probability density is given by Montgomery and Tidman (1964).

Let us now define the function $G(X_2, t_2|X_1, t_1)$ by

$$\begin{aligned}
G(X_2, t_2|X_1, t_1) &= \sum_{j=1}^{N} D_{j1}(X_2, t_2|X_1, t_1) \\
&= D_{11}(X_2, t_2|X_1, t_1) + (N-1)D_{21}(X_2, t_2|X_1, t_1) \\
&= D_{11}(X_2, t_2|X_1, t_1) + \frac{N-1}{V}\, g(X_2, t_2|X_1, t_1) + \frac{N-1}{V}\, f_1(X_2, t_2).
\end{aligned} \tag{6}$$

$G(X_2, t_2|X_1, t_1)$ may be interpreted as the conditional probability density of having any particle at (X_2, t_2) given particle 1 at (X_1, t_1). We consider the function

$$\begin{aligned}
\bar{G}(X_2, t_2|X_1, t_1) &= G(X_2, t_2|X_1, t_1) - \frac{N-1}{V}\, f_1(X_2, t_2) \\
&= D_{11}(X_2, t_2|X_1, t_1) + \frac{N-1}{V}\, g(X_2, t_2|X_1, t_1).
\end{aligned} \tag{7}$$

In (7) we see that, if $\bar{G}$ is expressed to the accuracy of zero order in $1/n_0$, we need to know D_{11} to the zero order and g to the first order. To the zero order, the particles are independent and their trajectories are rectilinear; therefore

$$D_{11}(X_2, t_2|X_1, t_1) = \delta(\boldsymbol{x}_2 - \boldsymbol{x}_1 - \boldsymbol{v}_1(t_2 - t_1))\delta(\boldsymbol{v}_2 - \boldsymbol{v}_1) \tag{8}$$

which satisfies the equation

$$\left(\frac{\partial}{\partial t_2} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{x}_2}\right) D_{11}(X_2, t_2|X_1, t_1) = 0. \tag{9}$$

$g(X_2, t_2|X_1, t_1)$ satisfies (5) to the first order. Substituting (7), (8) and (9) into (5), we have

$$\frac{\partial \bar{G}(X_2, t_2|X_1, t_1)}{\partial t_2} + \boldsymbol{v}_2 \cdot \frac{\partial \bar{G}(X_2, t_2|X_1, t_1)}{\partial \boldsymbol{x}_2}$$
$$= \frac{N-1}{mV} \frac{\partial f_1(X_2, t_2)}{\partial \boldsymbol{v}_2} \cdot \int \frac{\partial \phi(|\boldsymbol{x}_2 - \boldsymbol{x}_3|)}{\partial \boldsymbol{x}_2} \bar{G}(X_3, t_2|X_1, t_1)\, \mathrm{d}X_3. \tag{10}$$

Therefore the function $\bar{G}(X_2, t_2|X_1, t_1)$ satisfies the Vlasov equation.

Next, let us consider an integral I under the assumption that the initial correlation $g(X_2, 0|X_1, 0) = 0$. I is defined as

$$I(X_2, t_2, X_1, t_1) = \int \bar{G}(X_2, t_2|X, 0)\bar{G}(X, 0|X_1, t_1)\, \mathrm{d}X. \tag{11}$$

Obviously this integral satisfies the Vlasov equation (10). At $t_2 = 0$, (11) becomes

$$\begin{aligned} I(X_2, 0, X_1, t_1) &= \int \bar{G}(X_2, 0|X, 0)\bar{G}(X, 0|X_1, t_1)\, \mathrm{d}X \\ &= \int \delta(X_2 - X)\bar{G}(X, 0|X_1, t_1)\, \mathrm{d}X \\ &= \bar{G}(X_2, 0|X_1, t_1) \end{aligned} \tag{12}$$

since

$$\bar{G}(X_2, 0|X, 0) = D_{11}(X_2, 0|X, 0) + g(X_2, 0|X, 0) = \delta(X_2 - X_1). \tag{13}$$

Equation (12) shows that, at $t_2 = 0$, I assumes the proper initial condition $G(X_2, 0|X_1, t_1)$. Since it also satisfies the equation itself, I must be identical with $G(X_2, t_2|X_1, t_1)$. Therefore we have proved that

$$\bar{G}(X_2, t_2|X_1, t_1) = \int \bar{G}(X_2, t_2|X, 0)\bar{G}(X, 0|X_1, t_1)\, \mathrm{d}X \tag{14}$$

which is in the form of the conditional probability density of a Markov-like process.

The standard solution of (10) via a Fourier–Laplace transform can be written in the form (Montgomery and Tidman 1964)

$$\int \bar{G}(\boldsymbol{k}, \boldsymbol{v}_2, p|X, 0)\, \mathrm{d}\boldsymbol{v}_2 = \frac{1}{\epsilon^+(\boldsymbol{k}, p)} \int \frac{\bar{G}(\boldsymbol{k}, \boldsymbol{v}_2, 0|X, 0)}{p + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{v}_2}\, \mathrm{d}\boldsymbol{v}_2 \tag{15}$$

where $\bar{G}(\boldsymbol{k}, \boldsymbol{v}_2, p|X, 0)$ is the Fourier–Laplace transform of $\bar{G}(X_2, t|X, 0)$ in $\boldsymbol{x}_2$ and t. From (13) we have

$$\bar{G}(\boldsymbol{k}, \boldsymbol{v}_2, 0|X, 0) = \left(\frac{1}{2\pi}\right)^3 \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})\delta(\boldsymbol{v}_2 - \boldsymbol{v}_1). \tag{16}$$

Substituting (16) into (15), we have

$$\int \bar{G}(\boldsymbol{k}, \boldsymbol{v}_2, p|X, 0)\, \mathrm{d}\boldsymbol{v}_2 = \frac{1}{(2\pi)^3} \frac{1}{\epsilon^+(\boldsymbol{k}, p)} \frac{\exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})}{p + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{v}}. \tag{17}$$

The large-time behaviour of (17) can be obtained as

$$\int \bar{G}(\boldsymbol{k}, \boldsymbol{v}_2, t|X, 0)\,\mathrm{d}\boldsymbol{v}_2 = \frac{1}{(2\pi)^3}\frac{1}{\epsilon^+(\boldsymbol{k}, -\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{v})}\exp\{-\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{x}+\boldsymbol{v}t)\}. \tag{18}$$

3. Superposition principle

Consider any observables in the plasma of the form

$$\begin{aligned} A(\boldsymbol{x}_\mathrm{f}, t) &= \sum_{i=1}^{N} a(\boldsymbol{x}_\mathrm{f}, X_i(t)) \\ B(\boldsymbol{x}_\mathrm{f}, t) &= \sum_{i=1}^{N} b(\boldsymbol{x}_\mathrm{f}, X_i(t)) \end{aligned} \tag{19}$$

where a and b are first-order quantities (first order in discreteness parameters e, m, $1/n_0$, etc.) and the subscript f denotes field coordinates. Let us compute the correlation of the two observables to the first order:

$$\begin{aligned} &\langle A(\boldsymbol{x}_\mathrm{f}, t)B(\boldsymbol{x}_\mathrm{f}', t')\rangle - \langle A(\boldsymbol{x}_\mathrm{f}, t)\rangle\langle B(\boldsymbol{x}_\mathrm{f}', t')\rangle \\ &= \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}\langle a(\boldsymbol{x}_\mathrm{f}, X_i(t))b(\boldsymbol{x}_\mathrm{f}', X_j(t'))\rangle\right\} - \langle A(\boldsymbol{x}_\mathrm{f}, t)\rangle\langle B(\boldsymbol{x}_\mathrm{f}', t')\rangle \\ &= N\left\{\sum_{j=1}^{N}\langle a(\boldsymbol{x}_\mathrm{f}, X_i(t))b(\boldsymbol{x}_\mathrm{f}', X_j(t'))\rangle\right\} - \langle A(\boldsymbol{x}_\mathrm{f}, t)\rangle\langle B(\boldsymbol{x}_\mathrm{f}', t')\rangle \\ &= N\int\int a(\boldsymbol{x}_\mathrm{f}, X_1)b(\boldsymbol{x}_\mathrm{f}', X_2)G(X_2, t'|X_1, t)D_1(X_1, t)\,\mathrm{d}X_1\,\mathrm{d}X_2 \\ &\quad - N^2\int D_1(X_1, t)a(\boldsymbol{x}_\mathrm{f}, X_1)\,\mathrm{d}X_1\int D_1(X_2, t')b(\boldsymbol{x}_\mathrm{f}', X_2)\,\mathrm{d}X_2 \\ &= N\int a(\boldsymbol{x}_\mathrm{f}, X_1)b(\boldsymbol{x}_\mathrm{f}', X_2)\bar{G}(X_2, t'|X_1, t)\,\frac{1}{V}f_1(X_1, t)\,\mathrm{d}X_1\,\mathrm{d}X_2. \end{aligned} \tag{20}$$

Using (14) and the fact that

$$\bar{G}(X, 0|X_1, t)\,\frac{1}{V}f_1(X_1, t) = \bar{G}(X_1, t|X, 0)\,\frac{1}{V}f_1(X, 0) \tag{21}$$

equation (20) can be put in the form

$$\begin{aligned} &\langle A(\boldsymbol{x}_\mathrm{f}, t)B(\boldsymbol{x}_\mathrm{f}', t')\rangle - \langle A(\boldsymbol{x}_\mathrm{f}, t)\rangle\langle B(\boldsymbol{x}_\mathrm{f}', t')\rangle \\ &\quad = N\int \mathrm{d}X\frac{1}{V}f_1(X, 0)\left\{\int a(\boldsymbol{x}_\mathrm{f}, X_1)\bar{G}(X_1, t|X, 0)\,\mathrm{d}X_1\right\} \\ &\quad\quad \times\left\{\int b(\boldsymbol{x}_\mathrm{f}', X_2)\bar{G}(X_2, t'|X, 0)\mathrm{d}X_2\right\}. \end{aligned} \tag{22}$$

We define

$$\begin{aligned} \underline{X} &= (\boldsymbol{x}-\boldsymbol{v}t, \boldsymbol{v}) \\ \underline{X}' &= (\boldsymbol{x}'-\boldsymbol{v}'t', \boldsymbol{v}'). \end{aligned} \tag{23}$$

Using (13) and (14) together with the definition in (23), (22) can be written as

$$\begin{aligned} &\langle A(\boldsymbol{x}_\mathrm{f}, t)B(\boldsymbol{x}_\mathrm{f}', t')\rangle - \langle A(\boldsymbol{x}_\mathrm{f}, t)\rangle\langle B(\boldsymbol{x}_\mathrm{f}', t')\rangle \\ &= N\int \mathrm{d}X\;\mathrm{d}X'\,D_{11}(X, t; X', t')\langle a|\underline{X}, 0\rangle\langle b|\underline{X}', 0\rangle \end{aligned} \tag{24}$$

where

$$D_{11}(X, t; X', t') = Vf_1(X)\delta(x' - x - v'(t'-t))\delta(v' - v)$$

and is the zero-order joint probability density. Other quantities appearing in (24) are defined by

$$\langle a|\underline{X}, 0\rangle = \int a(x_f, X_1)\bar{G}(X_1, t|\underline{X}, 0)\,dX_1$$
$$\langle b|\underline{X}', 0\rangle = \int b(x_f', X_2)\bar{G}(X_2, t'|\underline{X}, 0)\,dX_2. \tag{25}$$

These are the quantities corresponding to the 'dressed particle' and (24) is the essential content of the superposition principle.

As an example, in the problem of Coulomb interactions between particles let us take $a(x_f, X_i)$ and $b(x_f', X_i)$ as the electric field. The interaction force is

$$\frac{\partial}{\partial x_f}\frac{e}{|x_f - x_i|}.$$

Either (22) or (24) can be used for the computation of the fluctuation of the electric field. For a homogeneous plasma the average values $\langle A\rangle$ and $\langle B\rangle$ in (22) or (24) vanish. Making use of the asymptotic time behaviour of $\int\bar{G}(k, v_2, t|X, 0)\,dv_2$ given in (18) and the Fourier transform of $(\partial/\partial x_f)(e/|x_f - x_i|)$, from the convolution theorem of the Fourier transform, (22) can be written as

$$\begin{aligned}\langle E(x_f, t)E(x_f', t')\rangle &= N\int f_1(v)\frac{dv}{V}\int dk\, ik\frac{e}{2\pi^2k^2}\frac{\exp(ik\,.\,x_f)\exp\{-ik\,.\,(x+vt)\}}{\epsilon^+(k, -ik\,.\,v)}\\ &\quad\times\int dk'\, ik'\frac{e}{2\pi^2k'^2}\frac{\exp(ik'\,.\,x_f')\exp\{-ik'\,.\,(x'+v't')\}}{\epsilon^+(k', -ik'\,.\,v')}\\ &= \frac{2n_0e^2}{\pi}\int dk\int_c\frac{dp}{2\pi i}\exp\{ik\,.\,(x_f'-x_f)\}\exp\{p(t'-t)\}\frac{k.k}{k^4}\\ &\quad\times\frac{1}{|\epsilon^+(k,p)|^2}\int f_1(v)\delta(p+ik\,.\,v)\,dv\end{aligned} \tag{26}$$

which is the fluctuation of the electric fields in a homogeneous plasma.

To compare our results with Rostoker's, let us multiply both sides of (14) by $f_1(X_1, t_1)$ and make use of the definition of $\bar{G}$ in (7) and the expression for D_{11} in (8). It is straightforward to show that

$$\begin{aligned}f_1(X_1, t_1)g(X_2, t_2|X_1, t_1) &= g(X_1, t_1|\underline{X}_2, 0)f_1(\underline{X}_2, 0) + g(X_2, t_2|\underline{X}_1, 0)f_1(\underline{X}_1, 0)\\ &\quad+ (N-1)\int dX\, dX'\, D_{11}(X, t; X', t')g(X_2, t_2|\underline{X}, 0)g(X_1, t_1|\underline{X}_2', 0).\end{aligned} \tag{27}$$

We note that our $g(X_2, t|\underline{X}_1, 0)$ corresponds to Rostoker's $P(X_1|X_2, t)$ (Rostoker 1964 b).

Finally, we point out that the above discussion can be generalized to the case of an inhomogeneous plasma. The main difference in the derivation is that for the inhomogeneous case, the zero-order trajectory of the particles is no longer rectilinear. Equation (9) becomes

$$\left\{\frac{\partial}{\partial t_2} + v_2\,.\,\frac{\partial}{\partial x_2} - \frac{n_0}{m}\int\frac{\partial\phi(|x_2 - x_3|)}{\partial x_2}f_1(X_3, t_2)\,dX_3\,.\,\frac{\partial}{\partial v_2}\right\}D_{11}(X_2, t_2|X_1, t_1) = 0 \tag{28}$$

with the solution

$$D_{11}(X_2, t_2|X_1, t_1) = \delta\big(X_1 - \psi(t_1, t_2, X_2)\big) \tag{29}$$

where $\psi(t, t_0, X_0)$ is the solution of the set of equations for the zero-order trajectory

$$\dot{\boldsymbol{x}} = \boldsymbol{v}$$
$$\dot{\boldsymbol{v}} = -\frac{n_0}{m}\int \frac{\partial\phi(|\boldsymbol{x}'-\boldsymbol{x}|)}{\partial \boldsymbol{x}'} f_1(X, t)\,\mathrm{d}X \tag{30}$$

with initial condition $\psi(t_0, t_0, X_0) = X_0$. Equations (14) and (24) can be derived for this case after some manipulations.

4. Conclusion

In this paper the superposition principle of 'dressed particles' in plasma kinetic theory is justified. While our results are closely related to Rostoker's (1964 a, b), the methods of derivation are quite different. Our starting point is the generalized stochastic equation. By introducing the conditional probability density $G(X_2, t_2|X_1, t_1)$ of finding any particle at (X_2, t_2) given a particle at (X_1, t_1), we are able to derive a Markov-type conditional probability density $\bar{G}(X_2, t_2|X_1, t_1)$ which is found to satisfy the Vlasov equation. With the aid of this probability density function, the superposition principle is justified without the explicit introduction of a test particle.

Acknowledgments

Discussions with K. C. Yeh of the University of Illinois are acknowledged with pleasure. This work was supported by NASA grant NsG 24.

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